International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Covariant Canonical Formalism of Fields

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Received August 5, 1997

The canonical formalism of fields consistent with the covariance principle of special relativity is given here. The covariant canonical transformations of fields are affected by 4-generating functions. All dynamical equations of fields, e.g., the Hamilton, Euler–Lagrange, and other field equations, are preserved under the covariant canonical transformations. The dynamical observables are also invariant under these transformations. The covariant canonical transformations are therefore fundamental symmetry operations on fields, such that the physical outcomes of each field theory must be invariant under these transformations. We give here also the covariant canonical equations of fields. These equations are the covariant versions of the Hamilton equations. They are defined by a density functional that is scalar under both the Lorentz and the covariant canonical transformations of fields.

1. INTRODUCTION

Particle fields are dictated by the covariance principle of the theory of relativity. This principle has been consistently implemented by the Lagrangian formalism of fields. The Hamiltonian formalism, on the other hand, is tacitly related to the time. This noncovariant aspect of the Hamiltonian formalism has, however, been extended to the whole canonical formalism of fields, such that the covariance and the canonical aspects of the fields have been foreign to each other.

One finds the inadequacy of the standard noncovariant canonical formalism of fields in the following observation. In classical particle dynamics, the canonical transformations are symmetry operations that leave the Hamilton equations invariant. The Hamiltonian, as an observable, is also invariant under these transformations. For fields, attention must also be paid to the invariance of Euler–Lagrange equations under the canonical transformations. Since the Lagrangian formalism is covariant, we see that the application of

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0020-7748/98/0300-78 5$15.00/0

noncovariant canonical transformations of fields is not adequate to preserve the Euler–Lagrange equations.

This feature has amounted to very limited applications of the canonical transformations to fields. We may summarize the situation by a quotation from a standard reference in the field: "There has been little exploration of canonical transformations for classical fields, a subject that for discrete systems proved to be so rich and consequential, (Goldstein, 1980, p. 567). A similar point of view has been taken by Merceir (1963).

In this article we undertake the problem of the covariant canonical formalism. We give the covariant canonical transformations of fields, and prove that these transformations are genuine symmetry operations on fields. Under the covariant canonical transformations all field equations and observables are invariant.

The generalized covariant versions of Hamilton equations are also given here. These covariant canonical equations are defined by a density functional that is scalar under both the Lorentz and the covariant canonical transformations of the fields.

The canonical transformations play a fundamental role in canonical gravity (Isham, 1993). They find also an important application in the related subject of spacetime dynamics (Mashkour, 1997). These issues have warranted the renewed interest in the subject.

The general conclusion of this article is summarized as follows: *the physical outcomes of each field theory must be invariant under the covariant canonical transformation of the fields* (Mashkour, 1997).

2. THE HAMILTONIAN FORMALISM

The Hamiltonian formalism shall be the guide for the development of the covariant canonical formalism of fields. Let $\{q^\alpha(x);\ \alpha = 1, 2, \ldots, N\}$ be the totality of the interacting fields. We assume the Lagrangian of these fields is of first order: $L = L(q, \partial q)$. The Hamiltonian density associated with this Lagrangian is then (Goldstein, 1980; Merceir, 1963)

$$H = \pi_\alpha \partial_0 q^\alpha - L \tag{1}$$

$\alpha = 1, 2, \ldots, N$ (summation over repeated indices is implied, unless stated otherwise). Variation of H yields

$$\delta H = \delta\pi_\alpha \partial_0 q^\alpha - \partial_0 \pi_\alpha \delta q^\alpha - \nabla \cdot (\mathbf{p}_\alpha \delta q^\alpha) \tag{2}$$

where

$$p_\alpha^\mu = \frac{\partial L}{\partial(\partial_\mu q^\alpha)} \tag{3}$$

We identify the variables p_α^μ, $\mu = 0, 1, 2, 3$, as the covariant 4-conjugate momentum of the field q^α. The above π_α is the time component of this momentum:

$$\pi_\alpha \equiv p_\alpha^0 \tag{4}$$

3. THE COVARIANT CANONICAL TRANSFORMATIONS

We demand that the covariant canonical transformations be symmetry operations on the dynamical equations. Therefore, the Hamilton and the Euler–Lagrange equations of the fields should be left invariant under these transformations. We consider the point canonical transformations which relate the field variables at the same spacetime point. We introduce the 4-generating functions:

$$F^\mu = F^\mu(q(\mathbf{x}, t), q'(\mathbf{x}, t)), \qquad \mu = 0, 1, 2, 3 \tag{5}$$

The more general case $F^\mu = F^\mu(q, q'; x)$ can be handled in a parallel manner. We write

$$L = \pi_\alpha \partial_0 q^\alpha - H \tag{6a}$$

$$= \pi'_\alpha \partial_0 q'^\alpha - H' + \partial_\mu F^\mu \tag{6b}$$

The second of the above equations defines the transformed set of variables (q'^α, π'_β) such that

$$\pi_\alpha(\mathbf{x}, t) = p_0^\alpha(\mathbf{x}, t) \tag{7a}$$

$$= \frac{\partial F^0}{\partial q^\alpha} \tag{7b}$$

$$\pi'_\alpha(\mathbf{x}, t) = p_0^\alpha(\mathbf{x}, t) \tag{7c}$$

$$= -\frac{\partial F^0}{\partial q'^\alpha} \tag{7d}$$

whereas

$$H' = H + \boldsymbol{\nabla} \cdot \mathbf{F} \tag{8}$$

Apart from the presence of the divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$ in equation (8), the above relations coincide with the canonical transformations of classical particle dynamics. These relations are normally extended to define the canonical transformations of fields (Goldstein, 1980; Merceir, 1963). In fact, under the transformations (7), the Hamilton equations for the fields are preserved.

Our objective is further to maintain under the canonical transformations the invariant forms for the Euler–Lagrange equations. This demands that the variation of H' take an identical form as δH of (2), i.e.,

$$\delta H' = \delta\pi'_\alpha \partial_0 q'^\alpha - \partial_0 \pi'_\alpha \delta q'^\alpha - \nabla \cdot (\mathbf{p}'_\alpha \delta q'^\alpha) \tag{9}$$

In fact, (2) and (9) are simultaneously satisfied if one makes the identification

$$\frac{\partial F^i}{\partial q^\alpha} = p^i_\alpha \tag{10a}$$

$$\frac{\partial F^i}{\partial q'^\alpha} = -p'^i_\alpha \tag{10b}$$

($i = 1, 2, 3$). At the same time it must be true that

$$\delta\pi'_\alpha \partial_0 q'^\alpha - \partial_0 \pi'_\alpha \delta q'^\alpha = \delta\pi_\alpha \partial_0 q^\alpha - \partial_0 \pi_\alpha \delta q^\alpha \tag{11}$$

We observe that (7) and (11) are identical to the canonical transformation relations of the classical particles dynamics. Therefore, (11) is known to be satisfied (Goldstein, 1980).

Equations (7) and (10) define the covariant canonical transformations of the canonical variables $\{q^\alpha, p^\mu_\alpha; \alpha = 1, 2, \ldots, N; \mu = 0, 1, 2, 3\}$. We show in Section 6 the invariance of the Euler–Lagrange equations under these transformations.

4. MISCELLANEOUS PROPERTIES AND ILLUSTRATIONS

1. We have, according to (11), that the set of canonical variables $\{q^\alpha, \pi_\beta \equiv p^0_\beta\}$ at a given point $(\mathbf{x}, t)$ define the same position–momentum Poisson bracket relations as in classical particles dynamics (Goldstein, 1980; Merceir, 1963):

$$\sum_\alpha \left(\frac{\partial \pi_\sigma}{\partial \pi'_\alpha} \frac{\partial q^\tau}{\partial q'^\alpha} - \frac{\partial q^\tau}{\partial \pi'_\alpha} \frac{\partial \pi_\sigma}{\partial q'^\alpha} \right) = \delta^\tau_\sigma \tag{12a}$$

$$\sum_\alpha \left(\frac{\partial q^\sigma}{\partial \pi'_\alpha} \frac{\partial q^\tau}{\partial q'^\alpha} - \frac{\partial q^\tau}{\partial \pi'_\alpha} \frac{\partial q^\sigma}{\partial q'^\alpha} \right) = 0 \tag{12b}$$

$$\sum_\alpha \left(\frac{\partial \pi_\tau}{\partial \pi'_\alpha} \frac{\partial \pi_\sigma}{\partial q'^\alpha} - \frac{\partial \pi_\sigma}{\partial \pi'_\alpha} \frac{\partial \pi_\tau}{\partial q'^\alpha} \right) = 0 \tag{12c}$$

2. The relations (7) and (10) take the combined covariant forms

$$p_\alpha^\mu = \frac{\partial F^\mu}{\partial q^\alpha}$$

$$= \frac{\partial L}{\partial(\partial_\mu q^\alpha)} \tag{13a}$$

$$p_\alpha'^\mu = -\frac{\partial F^\mu}{\partial q'^\alpha}$$

$$= \frac{\partial L'}{\partial(\partial_\mu q'^\alpha)} \tag{13b}$$

[the second line of (13b) is verified subsequently by (23a)].

3. One can define alternative generating functions as

$$\overline{F}^\mu(q, p'^\mu) \equiv F^\mu(q, q') + p_\alpha'^\mu q'^\alpha \tag{14}$$

where no summation is implies over the repeated index μ on the left-hand side. We have here

$$\delta\overline{F}^\mu = p_\alpha^\mu \delta q^\alpha - p_\alpha'^\mu \delta q'^\alpha + \delta p_\alpha'^\mu q'^\alpha + p_\alpha'^\mu \delta q'^\alpha \tag{15a}$$

$$= p_\alpha^\mu \delta q^\alpha + q'^\alpha \delta p_\alpha'^\mu \tag{15b}$$

$$\frac{\partial \overline{F}^\mu}{\partial q^\alpha} = p_\alpha^\mu \tag{15c}$$

$$\frac{\partial \overline{F}^\mu}{\partial p_\alpha'^\mu} = q'^\alpha \tag{15d}$$

The generating function $\overline{F}^\mu(q, p'^\mu)$ generalizes $F_2(q, p')$ of classical particle dynamics (Goldstein, 1980).

Illustrations. We give as examples the following generating functions:

$$\overline{F}_1^\mu = f^\alpha(q) p_\alpha'^\mu \tag{16a}$$

$$\overline{F}_2^\mu = f^\alpha(q) p_\alpha'^\mu + g^\mu(q) \tag{16b}$$

These generating functions correspond, respectively, to the transformations

$$q'^\alpha = f^\alpha(q), \qquad p_\alpha'^\mu = \frac{\partial q^\beta}{\partial q'^\alpha} p_\beta^\mu \tag{17a}$$

$$q'^\alpha = f^\alpha(q), \qquad p_\alpha'^\mu = \frac{\partial q^\beta}{\partial q'^\alpha} \left(p_\beta^\mu - \frac{\partial g^\mu}{\partial q^\beta} \right) \tag{17b}$$

The first of these transformations shows that a pure functional transformation of fields is a covariant canonical transformation.

5. THE POISSON BRACKET RELATIONS

In view of the covariant relations of (13), each of the pairings (q^{α}, p^{μ}_{β}), $\mu = 0, 1, 2, 3$, defines the same local Poisson bracket relations as in (12). These complete sets of brackets shall be given in a separate article in connection with the covariant Poisson bracket relations. We proceed here to develop further the Poisson bracket relations (12) associated with the pairing (q^{α}, $\pi_{\beta} = p^{0}_{\beta}$).

The Poisson bracket $\{\Pi, \Phi\}$ of two observables $\Pi(q(k), \pi(k))$ and $\Phi(q(l), \pi(l))$, at the spatial points (k) and (l), is defined as follows (Bjorken and Drell, 1965). We divide the space into infinitesimal cells of volumes τ_i and consider the fields and conjugate momenta at the different cells to be canonically independent. Then

$$\{\Pi(q(k), \pi(k)), \Phi(q(l), \pi(l))\}$$
$$= \lim_{\Delta \mathbf{x}_i \to 0} \sum_i \sum_\alpha \frac{1}{\tau_i} \left(\frac{\delta \Pi}{\delta \pi'_\alpha(i)} \frac{\delta \Phi}{\delta q'^{\alpha}(i)} - \frac{\delta \Phi}{\delta \pi'_\alpha(i)} \frac{\delta \Pi}{\delta q'^{\alpha}(i)} \right) \tag{18}$$

where $\Delta \mathbf{x}_i = (\Delta x^1_i, \Delta x^2_i, \Delta x^3_i)$ are the dimensions of the ith cell. Here $\{q'^{\alpha}(i), \pi'_\alpha(i)\}$ is an arbitrary canonical set of the fields and the time components of the 4-conjugate momenta at the spatial point (i). In fact, according to (12), the above bracket is invariant under the local canonical transformations.

One finds in particular that

$$\{\pi_\alpha(\mathbf{x}, t), q^{\beta}(\mathbf{x}', t)\} = \delta^{\beta}_{\alpha}\delta(\mathbf{x} - \mathbf{x}') \tag{19a}$$

$$\{\pi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{x}', t)\} = 0 \tag{19b}$$

$$\{q^{\alpha}(\mathbf{x}, t), q^{\beta}(\mathbf{x}', t)\} = 0 \tag{19c}$$

which are invariant under the local canonical transformations of (12).

6. THE INVARIANT LAGRANGE EQUATIONS

We come here to the conclusion that the Euler–Lagrange equations are invariant under the covariant canonical transformations (13). The transformed Lagrangian is, in fact, normally ignored on the particle mechanics level, since there the Hamiltonian and Lagrangian formalisms are completely equivalent. For fields, however, the Lagrangian formalism plays a wider role than the Hamiltonian formalism, where, for instance, the Lagrangian formalism defines the energy-momentum tensor and the related fields observables, while these variables cannot be obtained from the Hamiltonian formalism.

We would like, therefore, to define the Lagrangian of the canonically transformed fields and to establish the invariance of the Euler–Lagrange equations under the covariant canonical transformations. Let us rewrite (6b) as

$$H' = \pi'_\alpha \partial_0 q'^\alpha - (L - \partial_\mu F^\mu) \tag{20}$$

Comparison with (1) shows that the Lagrangian of the canonically transformed fields (q'^α) is

$$L' = \pi'_\alpha \partial_t q'^\alpha - H' \tag{21a}$$

$$= L(q, \partial q) - \partial_\mu F^\mu(q, q') \tag{21b}$$

One can vary either of the above right-hand side expressions to obtain the same field equations that govern the transformed fields (q'^α). Thus:

1. We consider, first, expression (21a), in which we use the already determined variation $\delta H'$ in (9). We find

$$\delta L' = \delta(\partial_0 q'^\alpha)\pi'_\alpha + \partial_0 q'^\alpha \delta\pi'_\alpha - (\partial_0 q'^\alpha)\delta\pi'_\alpha + \delta q'^\alpha \partial_0 \pi'_\alpha$$
$$+ \nabla \cdot (\mathbf{p}'_\alpha \delta q'^\alpha) \tag{22a}$$

$$= (\partial_\mu p'^\mu_\alpha)\delta q'^\alpha + p'^\mu_\alpha \delta(\partial_\mu q'^\alpha) \tag{22b}$$

from which it follows that

$$\frac{\partial L'}{\partial(\partial_\mu q'^\alpha)} = p'^\mu_\alpha \tag{23a}$$

$$\frac{\partial L'}{\partial q'^\alpha} = \partial_\mu p'^\mu_\alpha \tag{23b}$$

which yield

$$\partial_\mu \frac{\partial L'}{\partial(\partial_\mu q'^\alpha)} - \frac{\partial L'}{\partial q'^\alpha} = 0 \tag{24}$$

These are the desired Euler–Lagrange equations for the transformed fields (q'^α) in terms of the transformed Lagrangian L'.

2. The second alternative, equation (21b), yields even more information about the transformed Lagrangian and the field equations. It is also very instructive to give a detailed analysis of this alternative. By doing so we show explicitly the role of the 4-generating functions $(F^\mu(q, q'))$. One has from (21b) that

$$\delta L' = p^\mu_\alpha \delta(\partial_\mu q^\alpha) + \frac{\partial L}{\partial q^\alpha}\delta q^\alpha - p^\mu_\alpha \delta(\partial_\mu q^\alpha)$$
$$- (\delta p^\mu_\alpha)\partial_\mu q^\alpha + p'^\mu_\alpha(\delta\partial_\mu q'^\alpha) + (\delta p'^\mu_\alpha)\partial_\mu q'^\alpha$$
$$= \left(\frac{\partial L}{\partial q^\alpha} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu q^\alpha)}\right)\delta q^\alpha$$

$$+ \partial_\mu p_\alpha^\mu \delta q^\alpha - (\delta p_\alpha^\mu)\partial_\mu q^\alpha + (\delta p_\alpha'^\mu)\partial_\mu q'^\alpha + p_\alpha'^\mu(\delta \partial_\mu q'^\alpha) \quad (25)$$

We have that

$$\partial_\mu p_\alpha^\mu \delta q^\alpha = [(\partial^2 F^\mu / \partial q^\beta \partial q^\alpha)\partial_\mu q^\beta + (\partial^2 F^\mu / \partial q'^\beta \partial q^\alpha)\partial_\mu q'^\beta]\delta q^\alpha \quad (26a)$$

$$-(\delta p_\alpha^\mu)\partial_\mu q^\alpha = -[(\partial^2 F^\mu / \partial q^\beta \partial q^\alpha)\delta q^\beta + (\partial^2 F^\mu / \partial q'^\beta \partial q^\alpha)\delta q'^\beta]\partial_\mu q^\alpha \quad (26b)$$

$$(\delta p_\alpha'^\mu)\partial_\mu q'^\alpha = -[(\partial^2 F^\mu / \partial q^\beta \partial q'^\alpha)\delta q^\beta + (\partial^2 F^\mu / \partial q'^\beta \partial q'^\alpha)\delta q'^\beta]\partial_\mu q'^\alpha \quad (26c)$$

from which it follows that

$$\begin{aligned} &\partial_\mu p_\alpha^\mu \delta q^\alpha - \delta p_\alpha^\mu \partial_\mu q^\alpha + \delta p_\alpha'^\mu \partial_\mu q'^\alpha \\ &= -\left(\frac{\partial^2 F^\mu}{\partial q'^\beta \partial q^\alpha} \partial_\mu q^\alpha + \frac{\partial^2 F^\mu}{\partial q'^\beta \partial q'^\alpha} \partial_\mu q'^\alpha \right) \delta q'^\beta \\ &= \partial_\mu p_\beta'^\mu \delta q'^\beta \end{aligned} \quad (27)$$

Then (25) reduces to

$$\delta L' = \left[\frac{\partial L}{\partial q^\alpha} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu q^\alpha)} \right] \delta q^\alpha + \partial_\mu p_\alpha'^\mu \delta q'^\alpha + p_\alpha'^\mu \delta \partial_\mu q'^\alpha \quad (28)$$

Thus at the critical q^α of L where

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu q^\alpha)} - \frac{\partial L}{\partial q^\alpha} = 0 \quad (29)$$

we have that

$$\delta L' = \partial_\mu p_\alpha'^\mu \delta q'^\alpha + p_\alpha'^\mu \delta \partial_\mu q'^\alpha \quad (30)$$

which shows:

(a) The transformed Lagrangian is effectively also of the first order in the transformed fields

$$L' = L'(q', \partial_\mu q') \quad (31)$$

(b) Equation (30) in turn, as in alternative 1 above, yields

$$\partial_\mu \frac{\partial L'}{\partial(\partial_\mu q'^\alpha)} - \frac{\partial L'}{\partial q'^\alpha} = 0 \quad (32)$$

which together with (29) shows that the initial and the canonically transformed fields simultaneously satisfy the initial and the transformed Euler–Lagrange equations, respectively. Thus, the initial and transformed set of field equations and the respective solutions equivalently describe the physical situation.

7. THE INVARIANCE OF THE FIELD OBSERVABLES

We prove next that the field dynamical observables are, as well, invariant under the covariant local canonical transformations (13). The energy-momentum tensors of the initial fields (q^α) and the canonically transformed fields (q'^α) are, respectively,

$$T^\nu_\mu(q) = \frac{\partial L}{\partial(\partial_\nu q^\alpha)} \partial_\mu q^\alpha - \delta^\nu_\mu L \tag{33a}$$

$$T'^\nu_\mu(q') = \frac{\partial L'}{\partial(\partial_\nu q'^\alpha)} \partial_\mu q'^\alpha - \delta^\nu_\mu \nu L' \tag{33b}$$

Therefore, the total 4-momentum and angular momentum associated with the fields (q^α) and (q'^α) are

$$\bullet_q P_\mu = \int (\pi_\alpha \partial_\mu q^\alpha - \delta^0_\mu L)\, d\mathbf{x} \tag{34a}$$

$$\bullet_q J_{ij} = \int (x^i \pi_\alpha \partial_j q^\alpha - x^j \pi_\alpha \partial_i q^\alpha)\, d\mathbf{x} \tag{34b}$$

and

$$\bullet_{q'} P_\mu = \int (\pi'_\alpha \partial_\mu q'^\alpha - \delta^0_\mu L')\, d\mathbf{x} \tag{35a}$$

$$\bullet_{q'} J_{ij} = \int (x^i \pi'_\alpha \partial_j q'^\alpha - x^j \pi'_\alpha \partial_i q'^\alpha)\, d\mathbf{x} \tag{35b}$$

respectively. We wish to prove that

$$\bullet_{q'} P_\mu = \bullet_q P_\mu \tag{36a}$$

$$\bullet_{q'} J_{ij} = \bullet_q J_{ij} \tag{36b}$$

which would establish the invariance of the given observables under the covariant canonical transformation (13). In fact, we already have from (8) that

$$\begin{aligned} \bullet_{q'} P_0 &= \int H'\, d\mathbf{x} \\ &= \int (H + \nabla \cdot \mathbf{F})\, d\mathbf{x} \\ &= \bullet_q P_0 \end{aligned} \tag{37}$$

On the other hand, it follows from (7) that

$$\partial_i F^0 = \pi_\alpha \partial_i q^\alpha - \pi'_\alpha \partial_i q'^\alpha \tag{38}$$

Substituting this equation into (35) and integrating by parts, we conclude (36). This completes the proof that the given dynamical observables are invariant under the covariant canonical transformations (13).

8. THE COVARIANT CANONICAL EQUATIONS

We have thus established that the covariant canonical transformations of fields of (13) are genuine symmetry operations. They leave the Hamilton equations, the Euler–Lagrange equations, and the field observables invariant. We proceed to define the covariant canonical equations for the fields. The analysis shows further invariances under the covariant canonical transformations. We can write (21b) as

$$L' + \frac{\partial F^\mu}{\partial q'^\alpha} \frac{\partial q'^\alpha}{\partial x^\mu} = L - \frac{\partial F^\mu}{\partial q^\alpha} \frac{\partial q^\alpha}{\partial x^\mu} \tag{39}$$

which equivalently reads

$$L' - p'^\mu_\alpha \frac{\partial q'^\alpha}{\partial x^\mu} = L - p^\mu_\alpha \frac{\partial q^\alpha}{\partial x^\mu} \tag{40}$$

Let us then define the "scalar Hamiltonian" density

$$M = p^\mu_\alpha \frac{\partial q^\alpha}{\partial x^\mu} - L \tag{41}$$

We find

$$M' = M$$

Thus M is a scalar density under both the Lorentz and the covariant canonical transformations. We further have that

$$\delta M = \delta p^\mu_\alpha \partial_\mu q^\alpha - \partial_\mu p^\mu_\alpha \delta q^\alpha \tag{42a}$$

$$= \delta p'^\mu_\alpha \partial_\mu q'^\alpha - \partial_\mu p'^\mu_\alpha \delta q'^\alpha \tag{42b}$$

Thus

$$\frac{\partial M}{\partial p^\mu_\alpha} = \partial_\mu q^\alpha \tag{43a}$$

$$\frac{\partial M}{\partial q^\alpha} = -\partial_\mu p^\mu_\alpha \tag{43b}$$

Or equivalently

$$\frac{\partial M}{\partial p'^{\mu}_{\alpha}} = \partial_\mu q'^{\alpha} \tag{44a}$$

$$\frac{\partial M}{\partial q'^{\alpha}} = -\partial_\mu p'^{\mu}_{\alpha} \tag{44b}$$

Equations (43) and (44) are the covariant versions of the Hamilton equations. These equations now take simple differential forms, rather than the involved forms of Hamilton equations for the fields (Goldstein, 1980).

Equations (43) could also be obtained from a variational principle by treating all of the field variables (q^α, p^μ_α; $\mu = 0, 1, 2, 3$; $\alpha = 1, N$) as independent. The derivation goes parallel to the case of the particle Hamilton equations (Goldstein, 1980). It proves that the components of the 4-conjugate momentum (p^μ_α) have a completely symmetric canonical role.

8.1. An Illustration

We demonstrate the covariant canonical equations (43) by considering the case of the interacting electromagnetic fields with an external current j_μ. The Lagrangian density for this problem reads (Sakurai, 1978)

$$L = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + j_\mu A_\mu / c \tag{45}$$

We have that

$$p^\mu_\nu = \frac{\partial L}{\partial(\partial_\mu A_\nu)} \tag{46a}$$

$$= -(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{46b}$$

Here the subscript ν of p^μ_ν is the index of the conjugate field (A_ν), and the superscript μ is the component index of the 4-conjugate momentum. We have that

$$M = p^\mu_\nu \partial_\mu A^\nu - L \tag{47a}$$

$$= -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - j_\mu A_\mu / c \tag{47b}$$

This leads, according to (43b), to

$$\frac{\partial M}{\partial A_\nu} = -\frac{\partial p^\mu_\nu}{\partial x^\mu} \tag{48a}$$

$$= -j_\mu / c \tag{48b}$$

or

$$\partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = -j_\mu/c \tag{49}$$

which represents the desired Maxwell equations. It demonstrates the covariant canonical equations (43).

9. SUMMARY

We have put forward the theory of the covariant canonical formalism of fields. The covariant canonical transformations are defined by 4-generating functions $F^\mu(q, q')$, such that [see (21b)]

$$L(q, \partial q) \to L'(q', \partial q') = L(q, \partial q) - \partial_\mu F^\mu \tag{50}$$

whereas [see (13) and (23a)]

$$p^\mu_\alpha = \frac{\partial L}{\partial(\partial_\mu q^\alpha)} \tag{51a}$$

$$= \frac{\partial F^\mu}{\partial q^\alpha} \tag{51b}$$

and

$$p'^\mu_\alpha = \frac{\partial L'}{\partial(\partial_\mu q'^\alpha)} \tag{52a}$$

$$= -\frac{\partial F^\mu}{\partial q'^\alpha} \tag{52b}$$

Under the covariant canonical transformations all dynamical equations of the fields are preserved. The dynamical observables of the fields are, as well, invariant under these transformations. The covariant canonical transformations are therefore genuine symmetry operations on fields.

The generalized covariant Hamilton equations for fields also have been given. Here the canonical equations are defined in terms of a density functional that is scalar under both the Lorentz and the covariant canonical transformations. The generalized covariant canonical equations are thus fully invariant under the covariant canonical transformations.

Within the framework of the covariant canonical transformations, the equal-time Poisson bracket relations (19) are still satisfied. These relations correspond to the pairing (q^α, $\pi_\beta \equiv p^0_\beta$) of the fields and the time components of the 4-conjugate momenta. In a separate article we discuss the corresponding Poisson brackets corresponding to the pairings (q^α, p^i_β) of the fields (q^α) with each of the spatial components of the conjugate momenta (p^i_β). These four sets of Poisson bracket relations unite to define the covariant classical field

Poisson bracket relations. They correspond to the covariant commutation relations of the quantum fields.

ACKNOWLEDGMENTS

I wish to thank P. Badziag for reading the manuscript and for his very helpful remarks and suggestions. I also thank S. Tawfiq and M. Maqfi for discussions.

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